

CARMICHAEL NUMBERS AND THE SIEVE

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ABSTRACT. Using the sieve, we show that there are infinitely many Carmichael numbers whose prime factors all have the form $p = 1 + a^2 + b^2$ with $a, b \in \mathbb{Z}$.

*Dedicated to Carl Pomerance on the
occasion of his 70th birthday*

1. INTRODUCTION

For any prime number n , Fermat's little theorem asserts that

$$a^n \equiv a \pmod{n} \quad (a \in \mathbb{Z}). \quad (1.1)$$

Around 1910, Carmichael initiated the study of composite numbers n with the property (1.1); these are now known as *Carmichael numbers*. The existence of infinitely many Carmichael numbers was first established in the celebrated 1994 paper of Alford, Granville and Pomerance [1].

Since prime numbers and Carmichael numbers are linked by the common property (1.1), from a number-theoretic point of view it is natural to investigate various arithmetic properties of Carmichael numbers. For example, Banks and Pomerance [9] gave a conditional proof of their conjecture that there are infinitely many Carmichael numbers in an arithmetic progression $a + bc$ ($c \in \mathbb{Z}$) whenever $(a, b) = 1$. The conjecture was proved unconditionally by Matomäki [20] in the special case that a is a quadratic residue modulo b , and using an extension of her methods Wright [23] established the conjecture in full generality. The techniques introduced in [1] have led to many other investigations into the arithmetic properties of Carmichael numbers; see [2–5, 7, 8, 10, 14–19, 21, 24] and the references therein.

In this paper, we combine sieve techniques with the method of [1] to prove the following result.

THEOREM 1.1. *There exist infinitely many Carmichael numbers whose prime factors all have the form $p = 1 + a^2 + b^2$ with some $a, b \in \mathbb{Z}$. Moreover, there is a positive constant C such that the number of such Carmichael numbers not exceeding x is at least x^C (once x is sufficiently large in terms of C).*

REMARK 1.2. The Carmichael numbers described in this theorem seem to be quite unusual. Up to 10^8 , there are only seven such Carmichael numbers, namely

561, 162401, 410041, 488881, 656601, 2433601, 36765901.

By contrast, there are 255 “ordinary” Carmichael numbers up to 10^8 .

As is well known, whenever $p = 6k + 1$, $q = 12k + 1$ and $r = 18k + 1$ are simultaneously prime for some positive integer k , the number $n = pqr$ is a Carmichael number. However, no number of this form is a Carmichael number of the type described in the theorem, since $p - 1 = 6k$ and $r - 1 = 3 \cdot 6k$ cannot both be expressed as a sum of two squares. \square

Notation. Aside from notation introduced in situ, let \mathbb{P} be the set of primes, and let p and q always denote primes.

Let $a(b) := \{a + bc : c \in \mathbb{Z}\}$, $\mathbf{1}_S : \mathbb{N} \rightarrow \{0, 1\}$ the indicator function of $S \subseteq \mathbb{N}$, $\pi(x) := \sum_{n \leq x} \mathbf{1}_{\mathbb{P}}(n)$ and $\pi(x; b, a) := \sum_{n \leq x} \mathbf{1}_{\mathbb{P} \cap a(b)}(n)$. Let $\phi, \mu, \omega, P^+ : \mathbb{N} \rightarrow \mathbb{N}$ be the Euler, Möbius, number of distinct prime divisors and greatest prime divisor functions ($\omega(1) := 0$ and $P^+(1) := 1$). Let $\log_n : [1, \infty) \rightarrow [1, \infty)$ be the n th iterated logarithm, i.e., $\log_1 x := \max\{1, \log x\}$ and $\log_{n+1} x := \log_1(\log_n x)$.

Let expressions of the form $f(x) = O(g(x))$, $f(x) \ll g(x)$ and $g(x) \gg f(x)$ signify that $|f(x)| \leq c|g(x)|$ for all sufficiently large x , where $c > 0$ is an absolute constant. The notation $f(x) \asymp g(x)$ indicates that $f(x) \ll g(x) \ll f(x)$. We also let $f(x) = O_A(g(x))$ etc. have the same meanings with c depending on a parameter A . Finally, let $o_{x \rightarrow \infty}(1)$ (or simply $o(1)$ if x is clear in context) denote a quantity that tends to zero as x tends to infinity.

2. AGP SETUP

Let $\mathbb{B} := \{1, 5, 13, 17, 25, \dots\}$ be the multiplicative semigroup of the natural numbers generated by the set of primes $\mathbb{P} \cap 1(4)$, and let

$$\pi(x, y) := \#\{p \in \mathbb{B} \cap [2, x] : P^+(p-1) \leq y\}.$$

DEFINITION 2.1. Let \mathcal{E} be the set of numbers E in $(0, 1)$ for which there exist $x_1(E), \gamma_1(E) > 0$ such that for all $x \geq x_1(E)$, the inequality

$$\pi(x, x^{1-E}) \geq \gamma_1(E)\pi(x) \quad (2.1)$$

holds. \square

DEFINITION 2.2. Given $T \geq 3$, let $\ell(T)$ be the integer given in terms of putative Siegel zeros¹ in Lemma 3.1 below. \square

DEFINITION 2.3. For any fixed positive constants A, A' , let $\mathcal{B} = \mathcal{B}(A, A')$ denote the set of numbers $B \in (0, 1)$ for which the following holds. There exists $x_2(B)$ such that for all $x \geq x_2(B)$ we have

$$\frac{A^{-1}dx^{1-B}y^{-1}}{\phi(d)\log(dx^{1-B})} \leq \sqrt{\log x} \sum_{\kappa \leq x^{1-B}y^{-1}} \mathbf{1}_{\mathbb{B}}(\kappa) \mathbf{1}_{\mathbb{P}}(2d\kappa + 1) \leq \frac{A'dx^{1-B}y^{-1}}{\phi(d)\log(dx^{1-B})} \quad (2.2)$$

whenever $d \in \mathbb{B} \cap [1, x^B y]$, $|\mu(d)| = 1$, $P^+(d), y \leq x^{B/\log_2 x}$ and $(d, \ell(x^B)) = 1$. \square

Matomäki [20, Lemma 2] has shown that $\mathcal{E} \supseteq (0, 1/2)$. By Lemma 3.4 below, if A, A' are sufficiently large² and β is sufficiently small (depending on A, A'),

¹We take license with the term ‘‘Siegel zero’’ — cf. Lemma 3.1 below for a precise statement.

²Although we do not give details, one can show that $A = 50$ and $A' = 1$ suffice. We do not compute a value for β .

then $\mathcal{B} \supseteq (0, \beta)$. Consequently, the following analogue of [1, Theorem 4.1] immediately implies Theorem 1.1. Its proof relies on Lemma 2.5 below, which is itself analogous to [1, Theorem 3.1].

THEOREM 2.4. *Let $C(x)$ denote the number of Carmichael numbers up to x all of whose prime divisors p are such that $(p-1)/2 \in \mathbb{B}$. For each $E \in \mathcal{E} \cap (4/9, 1)$, $B \in \mathcal{B}$ and $\epsilon > 0$, there is a number $x_4(E, B, \epsilon)$, such that whenever $x \geq x_4(E, B, \epsilon)$, we have $C(x) \geq x^{EB-\epsilon}$.*

LEMMA 2.5. *Fix any $B \in \mathcal{B}$. There exists $x_3(B)$ such that the following holds for all $x \geq x_3(B)$ and any integer L satisfying hypotheses (H1) — (H5) below. There is some $k \in [1, x^{1-B}] \cap \mathbb{B}$ with $(k, L) = 1$ such that*

$$4A(\log x)^{3/2} \sum_{d|L, 2dk+1 \leq x} \mathbf{1}_{\mathbb{P}}(2dk+1) > \# \{d \mid L : d \leq x^B\}.$$

Our hypotheses are the following:

- (H1) $L \in \mathbb{B}$ and $|\mu(L)| = 1$;
- (H2) $P^+(L) \leq x^{B/\log_2 x}$;
- (H3) $(L, \ell(x^B)) = 1$;
- (H4) for any $d \mid L$ with $d \leq x^B$, the bound $16A\sqrt{\log x} \sum_{q|d} 1/q \leq 1 - B$ holds;
- (H5) we have $24AA' \sum_{q|L} 1/q \leq 5(1 - B)$.

Proof. Let $x \geq x_3(B)$ with $x_3(B)$ sufficiently large (to be specified). We have

$$\sum_{\substack{\kappa \leq x^{1-B} \\ (\kappa, L)=1}} \mathbf{1}_{\mathbb{B}}(\kappa) \sum_{d|L, d \leq x^B} \mathbf{1}_{\mathbb{P}}(2d\kappa+1) = \sum_{d|L, d \leq x^B} \sum_{\substack{\kappa \leq x^{1-B} \\ (\kappa, L)=1}} \mathbf{1}_{\mathbb{B}}(\kappa) \mathbf{1}_{\mathbb{P}}(2d\kappa+1),$$

so there must be some $k \in [1, x^{1-B}] \cap \mathbb{B}$ with $(k, L) = 1$ for which

$$x^{1-B} \sum_{d|L, d \leq x^B} \mathbf{1}_{\mathbb{P}}(2dk+1) \geq \sum_{d|L, d \leq x^B} \sum_{\substack{\kappa \leq x^{1-B} \\ (\kappa, L)=1}} \mathbf{1}_{\mathbb{B}}(\kappa) \mathbf{1}_{\mathbb{P}}(2d\kappa+1). \quad (2.3)$$

Let $d \mid L, d \leq x^B$. Note that d is squarefree, $P^+(d) \leq x^{B/\log_2 x}$ and $(d, \ell(x^B)) = 1$. Observe that

$$\begin{aligned} & \sum_{\substack{\kappa \leq x^{1-B} \\ (\kappa, L)=1}} \mathbf{1}_{\mathbb{B}}(\kappa) \mathbf{1}_{\mathbb{P}}(2d\kappa+1) \\ & \geq \sum_{\kappa \leq x^{1-B}} \mathbf{1}_{\mathbb{B}}(\kappa) \mathbf{1}_{\mathbb{P}}(2d\kappa+1) - \sum_{q|L} \sum_{mq \leq x^{1-B}} \mathbf{1}_{\mathbb{B}}(mq) \mathbf{1}_{\mathbb{P}}(2dmq+1). \end{aligned} \quad (2.4)$$

We can assume that $x_3(B) \geq x_2(B)$; hence by (2.2) we have

$$A\sqrt{\log x} \sum_{\kappa \leq x^{1-B}} \mathbf{1}_{\mathbb{B}}(\kappa) \mathbf{1}_{\mathbb{P}}(2d\kappa+1) \geq \frac{dx^{1-B}}{\phi(d) \log(dx^{1-B})} \geq \frac{dx^{1-B}}{\phi(d) \log x}. \quad (2.5)$$

Now fix $q \mid L$ for the moment, and consider the sum on $mq \leq x^{1-B}$ in (2.4). Note that

$$\sum_{mq \leq x^{1-B}} \mathbf{1}_{\mathbb{B}}(mq) \mathbf{1}_{\mathbb{P}}(2dmq+1) \leq \pi(2dx^{1-B} + 1; dq, 1) \leq \pi(2dx^{1-B}; dq, 1) + 1.$$

The Brun–Titchmarsh inequality of Montgomery and Vaughan [22] gives

$$\pi(dx^{1-B}; 2dq; 1) < \frac{4dx^{1-B}}{\phi(dq) \log(x^{1-B}/q)} \leq \frac{8}{q(1-B)} \frac{dx^{1-B}}{\phi(d) \log x} - 1,$$

provided $x_3(B)$ is sufficiently large, which we assume (recall that $q \leq x^{B/\log_2 x}$). Using (H4) it follows that

$$2A\sqrt{\log x} \sum_{q|d} \sum_{mq \leq x^{1-B}} \mathbf{1}_{\mathbb{B}}(mq) \mathbf{1}_{\mathbb{P}}(2dmq + 1) < \frac{dx^{1-B}}{\phi(d) \log x}. \quad (2.6)$$

Now suppose $q \nmid d$. For such q we have $dq \mid L$, $|\mu(dq)| = 1$, $P^+(dq) \leq x^{B/\log_2 x}$; therefore, applying (2.2) ($d \mapsto dq$, $y \mapsto q$) and noting that $q/\phi(q) \leq 6/5$ for all $q \geq 5$, it follows that

$$\sqrt{\log x} \sum_{m \leq x^{1-B}/q} \mathbf{1}_{\mathbb{B}}(m) \mathbf{1}_{\mathbb{P}}(2(dq)m + 1) \leq \frac{A'dx^{1-B}}{\phi(dq) \log(dqx^{1-B})} \leq \frac{6A'}{5q(1-B)} \frac{dx^{1-B}}{\phi(d) \log x}.$$

Since $\mathbf{1}_{\mathbb{B}}(mq) = \mathbf{1}_{\mathbb{B}}(m)$ we deduce from (H5) that

$$4A\sqrt{\log x} \sum_{q|L, q \nmid d} \sum_{mq \leq x^{1-B}} \mathbf{1}_{\mathbb{B}}(mq) \mathbf{1}_{\mathbb{P}}(2dmq + 1) \leq \frac{dx^{1-B}}{\phi(d) \log x}. \quad (2.7)$$

Combining (2.4) – (2.7) we see that

$$4A\sqrt{\log x} \sum_{\substack{\kappa \leq x^{1-B} \\ (\kappa, L)=1}} \mathbf{1}_{\mathbb{B}}(\kappa) \mathbf{1}_{\mathbb{P}}(2d\kappa + 1) > \frac{dx^{1-B}}{\phi(d) \log x} (4 - 2 - 1) \geq \frac{x^{1-B}}{\log x},$$

and combining this with (2.3) we obtain the stated result. \square

Proof of Theorem 2.4. Minor modifications notwithstanding, the proof follows that of [1, Theorem 4.1] verbatim, so let us only set up the proof here. Let $E \in \mathcal{E}$, $B \in \mathcal{B}$, $\epsilon > 0$. We can assume that $\epsilon < EB$. Let $\theta := (1 - E)^{-1}$ and let $y \geq 2$ be a parameter. Put

$$\mathcal{Q} := \{q \in \mathbb{B} \cap (y^\theta / \log y, y^\theta] : P^+(q - 1) \leq y\},$$

and let ℓ be a positive integer (to be specified) satisfying $\log \ell \ll y^\theta / \log y$. By (2.1) we have

$$|\mathcal{Q} \setminus \{q : q \mid \ell\}| \geq \frac{1}{2} \gamma_1(E) \frac{y^\theta}{\log(y^\theta)}$$

for all large y (we have $\pi(y^\theta / \log y) \ll y^\theta / (\log(y^\theta) \log y)$ using Chebyshev's bound, as well as the well-known bound $\omega(\ell) \ll (\log \ell) / (\log_2 \ell)$). Let $L := \prod_{q \in \mathcal{Q}, q \nmid \ell} q$; then

$$\log L \leq |\mathcal{Q}| \log(y^\theta) \leq \pi(y^\theta) \log(y^\theta) \leq 2y^\theta$$

for all large y . Let $\delta := \epsilon\theta / (4B)$ and let $x := e^{y^{1+\delta}}$. We have

$$\sum_{q|L} \frac{1}{q} \leq \sum_{q \in (y^\theta / \log y, y^\theta]} \frac{1}{q} \leq 2 \frac{\log_2 y}{\theta \log y} \leq \frac{5(1-B)}{24AA'}$$

for all sufficiently large y . For any $d \mid L$ with $d \leq x^B$ we have $\omega(d) \leq 2 \log x / \log_2 x$ (if x is large enough), and therefore

$$\sum_{q \mid d} \frac{1}{q} \leq \frac{\log y}{y^\theta} \frac{2 \log x}{\log_2 x} < \frac{2 \log x}{(\log x)^{\theta/(1+\delta)}} < \frac{1-B}{16A\sqrt{\log x}}$$

for all large y provided that $\theta/(1+\delta) > 3/2$. Since

$$4\delta = \epsilon\theta/B < \theta E = E/(1-E),$$

we have

$$2\theta - 3\delta = 2\left(1 + \frac{E}{1-E}\right) - 3\delta > 2\left(1 + \frac{E}{1-E}\right) - \frac{3E}{4(1-E)} = 2 + \frac{5E}{4(1-E)},$$

and this is greater than three (and hence $\theta/(1+\delta) > 3/2$ as required) whenever $5E/(4(1-E)) > 1$, i.e., $E > 4/9$, which we assume.

We now specify that $\ell := \ell(x^B)$. We clearly have $\ell(x^B) \leq x^B$ (cf. Lemma 3.1), so the requirement that $\log \ell \ll y^\theta / \log y$ is satisfied:

$$\log \ell \leq \log x = y^{1+\delta} < y^{2\theta/3} \ll y^\theta / \log y.$$

We can apply Lemma 2.5 with $B, x, L, \ell = \ell(x^B)$. Thus, for all sufficiently large values of y , there is an integer $k \in \mathbb{B}$ coprime to L , for which the set \mathcal{P} of primes $p \leq x$ with $p = 2dk + 1$ for some divisor d of L , satisfies

$$|\mathcal{P}| \geq \frac{\#\{d \mid L : d \leq x^B\}}{4A(\log x)^{3/2}}.$$

We leave the reader to pursue the remainder of the proof in [1]. \square

3. THE SIEVE

Notational caveat. This section can be read independently of §2, and below A, B, d, k are not the same as in §2.

Level of distribution. We first quote part of [6, Lemma 4.1], which gives a qualitative extension of the classical (exceptional) zero-free region for Dirichlet L -functions in the case of smooth moduli. Its proof uses bounds for character sums to smooth moduli due to Chang [11].

LEMMA 3.1. *Let $T \geq 3$. Among all primitive Dirichlet characters $\chi \bmod \ell$ to moduli ℓ satisfying $\ell \leq T$ and $P^+(\ell) \leq T^{1/\log_2 T}$, there is at most one for which the associated L -function $L(s, \chi)$ has a zero in the region*

$$\Re(s) > 1 - c \log_2 T / \log T, \quad |\Im(s)| \leq \exp(\sqrt{\log T} / \log_2 T), \quad (3.1)$$

where $c > 0$ is a certain (small) absolute constant. If such a character $\chi \bmod \ell$ exists, then χ is real and $L(s, \chi)$ has just one zero in the region (3.1), which is real and simple, and we set $\ell(T) := \ell$. Otherwise we set $\ell(T) := 1$.

REMARK 3.2. If $\chi \bmod \ell$ is real and primitive, then $\ell = 2^\nu \hat{\ell}$ where $\nu \leq 3$ and $\hat{\ell}$ is odd and squarefree. By Siegel's theorem [12, §21, (4)], if β is any real zero of $L(s, \chi)$ then $\ell \gg_A (1 - \beta)^{-A}$ for any $A > 1$. Hence, if $\ell = \ell(T)$ is as in Lemma 3.1 and $\ell \neq 1$, then

$$\ell \gg_A (\log \ell / \log_2 \ell)^A. \quad (3.2)$$

The implicit constant is ineffective for $A > 2$, but it is effective for $2 \geq A > 1$, and consequently the implicit constant in (3.3) below is effective for $A < 2$. \square

The following statement is a consequence of [6, Theorem 4.1], whose proof combines standard zero density estimates with the zero free region for smooth moduli given in Lemma 3.1.

THEOREM 3.3. *Fix $\eta > 0$. Let $x \geq 3^{1/\eta}$ be a number, and let $k \geq 1$ be a squarefree integer such that $P^+(k) < x^{\eta/\log_2 x}$, $k < x^\eta$ and $(k, \ell) = 1$, where $\ell := \ell(x^\eta)$ as in Lemma 3.1. If $\eta = \eta(A, \delta)$ is sufficiently small in terms of any fixed $A > 0$ and $\delta \in (0, 1/2)$, then*

$$\sum_{r \leq \sqrt{x}/x^\delta} \max_{(a, kr)=1} \left| \pi(x; kr, a) - \frac{\pi(x)}{\phi(kr)} \right| \ll_{\delta, A} \frac{x}{\phi(k)(\log x)^A}. \quad (3.3)$$

Proof. Let us write $\Delta(x; kr, a)$ for $\pi(x; kr, a) - \pi(x)/\phi(kr)$. The bound

$$\sum_{\substack{r \leq \sqrt{x}/x^\delta \\ (r, P^+(\ell))=1}} \max_{(a, kr)=1} |\Delta(x; kr, a)| \ll_{\delta, A} \frac{x}{\phi(k)(\log x)^A} \quad (3.4)$$

is³ [6, Theorem 4.1] in our notation, except that we have the stronger hypothesis that $(k, \ell) = 1$, whereas in [6] it is only assumed that $(k, P^+(\ell)) = 1$. If $\ell = 1$ then we are done, so let us assume $\ell \neq 1$. By Remark 3.2, $\ell = 2^\nu \hat{\ell}$, where $\nu \leq 3$ and $\hat{\ell}$ is a product of $O(\log x^\eta / \log_2 x^\eta)$ distinct odd primes. The bound (3.4) holds if $P^+(\ell)$ is replaced by any prime divisor of ℓ , as is manifest from the proof of [6, Theorem 4.1] (the crux being that $\ell \nmid r$). Summing over the prime divisors of $\hat{\ell}$, replacing A by $A + 1$ in (3.4), and recalling that η depends only on A and δ , we deduce that

$$\sum_{\substack{r \leq \sqrt{x}/x^\delta \\ \hat{\ell} \nmid r}} \max_{(a, kr)=1} |\Delta(x; kr, a)| \ll_{\delta, A} \frac{x}{\phi(k)(\log x)^A}. \quad (3.5)$$

On the other hand, using $\pi(x) \ll x/\log x$ together with the Brun–Titchmarsh inequality [13, (13.3) et seq.] we obtain that, uniformly for $r \leq \sqrt{x}$ with $\hat{\ell} \mid r$ and $(a, kr) = 1$,

$$\Delta(x; kr, a) \ll \frac{x}{\phi(kr) \log x}.$$

For any such r , write $r = \hat{\ell} r_1 r_2$, where r_1 is composed of primes dividing ℓ , and $(r_2, \ell) = 1$. Note that $r_1 \leq \sqrt{x}/(r_2 \hat{\ell})$, $(kr_2, \hat{\ell}) = 1$ (since $(k, \ell) = 1$), and $\phi(kr) \geq \phi(k)\phi(\hat{\ell})\phi(r_1)$; therefore,

$$\sum_{\substack{r \leq \sqrt{x}/x^\delta \\ \hat{\ell} \mid r}} \max_{(a, kr)=1} |\Delta(x; kr, a)| \ll \frac{x}{\phi(k)\phi(\hat{\ell}) \log x} \sum_{r_1 \leq \sqrt{x}} \frac{1}{\phi(r_1)} \ll \frac{x}{\phi(k)\phi(\hat{\ell})}. \quad (3.6)$$

³Actually, in [6, Theorem 4.1] the primes are counted with a logarithmic weight, from which one can deduce, via partial summation, the bound as stated in (3.4), and this is the form in which the bound is ultimately used in [6].

Since $\ell/\phi(\ell) \ll \log_2 \ell \ll \log_2 x^\eta$ and $\ell \gg_A (\log x^\eta / \log_2 x^\eta)^A$ by (3.2), we see that

$$1/\phi(\hat{\ell}) \ll (\log_2 x^\eta)^{A+1}/(\log x^\eta)^A,$$

thus combining (3.5) with (3.6) gives the result (with A replaced by any smaller constant). \square

Setup & key estimate. Equipped with our level of distribution result, establishing our key estimate involves a routine application of the semi-linear sieve and a “switching trick” (as in [13, Theorem 14.8]). We are to sieve a sequence of primes in arithmetic progression by the primes in $\mathbb{P} \cap 3(4)$.

For $x \geq 3$, let

$$P(x) := \prod_{\substack{p < x \\ p \equiv 3(4)}} p,$$

let

$$V(x) := \prod_{\substack{p < x \\ p \equiv 3(4)}} \left(1 - \frac{1}{\phi(p)}\right) = \prod_{\substack{p < x \\ p \equiv 3(4)}} \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p(p-2)}\right)^{-1} \quad (3.7)$$

and let

$$W(x) := \prod_{\substack{p < x \\ p \equiv 1(4)}} \left(1 + \frac{1}{\phi(p)} + \frac{1}{\phi(p^2)} + \cdots\right) = \prod_{\substack{p < x \\ p \equiv 1(4)}} \left(1 - \frac{1}{p}\right)^{-1} \left(1 + \frac{1}{p(p-1)}\right).$$

For future reference, we record here that by Mertens’ theorem one has

$$W(x)/V(x) = \frac{1}{2}A_1A_3e^\gamma \log x + O(1), \quad (3.8)$$

where

$$A_1 := \prod_{p \equiv 1(4)} \left(1 + \frac{1}{p(p-1)}\right) \quad \text{and} \quad A_3 := \prod_{p \equiv 3(4)} \left(1 + \frac{1}{p(p-2)}\right). \quad (3.9)$$

By Mertens’ theorem we also have, for $2 \leq x < y$ and $j = 1, 3$,

$$\sum_{\substack{x \leq p < y \\ p \equiv j(4)}} \frac{1}{p} = \frac{1}{2} \log \left(\frac{\log y}{\log x} \right) + O \left(\frac{1}{\log x} \right) \leq \frac{\log(y/x)}{2 \log x} \left(1 + O \left(\frac{1}{\log(y/x)} \right) \right), \quad (3.10)$$

and furthermore,

$$\frac{V(x)}{V(y)}, \frac{W(y)}{W(x)} = \left(\frac{\log x}{\log y} \right)^{1/2} \left(1 + O \left(\frac{1}{\log y} \right) \right). \quad (3.11)$$

Indeed, we actually have (cf. [13, (14.21)–(14.24)])

$$1/V(x) = 2A_3B\sqrt{(e^\gamma/\pi) \log x} (1 + O(1/\log x)) \quad (3.12)$$

and

$$W(x) = (\pi A_1/4B)\sqrt{(e^\gamma/\pi) \log x} (1 + O(1/\log x)),$$

where

$$B := \frac{1}{\sqrt{2}} \prod_{p \equiv 3(4)} \left(1 - \frac{1}{p^2}\right)^{-1/2} = \frac{\pi}{4} \prod_{p \equiv 1(4)} \left(1 - \frac{1}{p^2}\right)^{1/2} = 0.764223 \dots$$

is the Landau–Ramanujan constant. Finally, let $f(s)$ and $F(s)$ be the continuous solutions to the following system of differential-difference equations:

$$\begin{aligned} \sqrt{s}F(s) &= 2\sqrt{e^\gamma/\pi} & (0 \leq s \leq 2), & & (\sqrt{s}F(s))' &= f(s-1)/2\sqrt{s} & (s > 0) \\ f(1) &= 0 & & & (\sqrt{s}f(s))' &= F(s-1)/2\sqrt{s} & (s > 1). \end{aligned}$$

For $1 \leq s \leq 3$ we have [13, p.275]

$$\frac{\sqrt{s}f(s)}{\sqrt{e^\gamma/\pi}} = \int_1^s \frac{dt}{\sqrt{t(\log t)}} = \log \left(1 + 2(s-1) + 2\sqrt{s(s-1)} \right). \quad (3.13)$$

LEMMA 3.4. Fix $\eta > 0$. Let $x \geq 3^{1/\eta}$ be a number, and let $k \geq 1$ be a squarefree integer, such that $P^+(k) < x^{\eta/\log_2 x}$, $k < x^\eta$ and $(k, \ell) = 1$, with $\ell := \ell(x^\eta)$ as in Lemma 3.1. If $k \in \mathbb{B}$ and η is sufficiently small, then

$$\sum_{m \leq x} \mathbf{1}_{\mathbb{B}}(m) \mathbf{1}_{\mathbb{P}}(2km+1) \asymp \frac{kx}{\phi(k)(\log x)^{3/2}}. \quad (3.14)$$

Proof. Let $k \in \mathbb{B}$ be fixed. Note that $(k, 2P(x)) = 1$. As $\mathbf{1}_{\mathbb{B}}(m) = 1$ implies that $m \equiv 1 \pmod{4}$, and thus $2km+1 \equiv 3 \pmod{8}$, we can assume that our sum is over m for which $2km+1 \equiv j \pmod{8k}$ for some reduced residue $j \pmod{8k}$, with $j \equiv 3 \pmod{8}$ and $j \equiv 1 \pmod{k}$. Thus, we want to sift the sequence $\mathcal{A} := (\mathbf{1}_{\mathbb{P} \cap j \pmod{8k}}(2km+1))$ by the primes in $\mathbb{P} \cap 3 \pmod{4}$, and the sum in (3.14) is equal to $S(\mathcal{A}, \sqrt{x})$, where

$$S(\mathcal{A}, z) := \sum_{\substack{m \leq x \\ (m, P(z))=1}} \mathbf{1}_{\mathbb{P} \cap j \pmod{8k}}(2km+1)$$

is our sifting function.

Let $z < x$. Suppose $d \mid P(z)$ and note that $(d, 2k) = 1$ (since $2 \nmid P(z)$ and $(k, P(z)) = 1$). Thus, $d \mid m$ if and only if $2km+1 \equiv 1 \pmod{d}$, and so

$$\mathcal{A}_d(x) := \sum_{\substack{m \leq x \\ d \mid m}} \mathbf{1}_{\mathbb{P} \cap j \pmod{8k}}(2km+1) = \pi(2kx+1; 8dk, h) = g(d)X + r_d$$

for some reduced residue $h \pmod{8dk}$ with $h \equiv j \pmod{8k}$ and $h \equiv 1 \pmod{d}$, and where $X := \pi(2kx)/\phi(8k)$, $g(d) := 1/\phi(d)$ and

$$r_d := \mathcal{A}_d(x) - g(d)X = \pi(2kx+1; 8dk, h) - \pi(2kx)/\phi(8dk).$$

Now set $\delta := 1/3890$. (The argument below works for any sufficiently small δ .) By Theorem 3.3 ($x \mapsto 2kx$) our sequence \mathcal{A} has level of distribution $D := \sqrt{x}/x^\delta$, and we have

$$R(D, z) := \sum_{d \mid P(z), d < D} |r_d| \ll_\delta X (\log x)^{-2/3}$$

provided that $\eta = \eta(\delta)$ is sufficiently small, which we assume. We fix our sifting level z and sifting variable s at

$$z := D/x^\delta = \sqrt{x}/x^{2\delta} \quad \text{and} \quad s := \log D / \log z = (1-2\delta)/(1-4\delta) = 1944/1943.$$

We can infer from (3.11) and [13, Theorem 11.12–Theorem 11.13 et seq.] that

$$S(\mathcal{A}, z) \geq XV(z) \{f(s) + O((\log D)^{-1/6})\} - R(D, z),$$

and

$$S(\mathcal{A}, z) \leq XV(z) \{F(s) + O((\log D)^{-1/6})\} + R(D, z).$$

As $V(z) \asymp (\log z)^{1/2}$ by (3.12) and $R(D, z) \ll_\delta X(\log z)^{2/3}$, the latter can be subsumed under the O -term in each case, hence

$$f(s) + O_\delta((\log x)^{-1/6}) \leq \frac{S(\mathcal{A}, z)}{XV(z)} \leq F(s) + O_\delta((\log x)^{-1/6}). \quad (3.15)$$

Since $S(\mathcal{A}, \sqrt{x}) \leq S(\mathcal{A}, z)$, the upper bound in (3.14) follows. We claim that

$$\frac{S(\mathcal{A}, z) - S(\mathcal{A}, \sqrt{x})}{XV(z)} \leq \frac{1}{2}f(s) + O_\delta((\log x)^{-1/6}), \quad (3.16)$$

which, when combined with the first inequality in (3.15), gives the lower bound in (3.14).

For $z < \sqrt{x}$ we have Buchstab's identity [13, (6.4)]:

$$S(\mathcal{A}, z) - S(\mathcal{A}, \sqrt{x}) = \sum_{\substack{z < p_1 \leq \sqrt{x} \\ p_1 \equiv 3 \pmod{4}}} \sum_{\substack{m \leq x \\ p_1 | m \\ (m, P(p_1)) = 1}} \mathbf{1}_{\mathbb{P} \cap j(8k)}(2km + 1) =: T.$$

Suppose $x^{1/3} \leq z < \sqrt{x}$ and consider any m that makes a nonzero contribution to the inner sum in T . We have $p_1 \mid m$ and $m \leq p_1^3$ for some $p_1 \equiv 3 \pmod{4}$, m is not divisible by any prime less than p_1 in $\mathbb{P} \cap 3 \pmod{4}$, yet recall that $m \equiv 1 \pmod{4}$ (for $2km + 1 \equiv j \equiv 3 \pmod{8}$ and $k \equiv 1 \pmod{4}$). Therefore, $p_2 \mid m$ for exactly one prime $p_2 \equiv 3 \pmod{4}$ in addition to p_1 . Since $(k, p_1 p_2) = 1$, we conclude that $m = ap_1 p_2$ for some a, p_1, p_2 such that

$$a \equiv 1 \pmod{4}, \quad p_1 \equiv p_2 \equiv 3 \pmod{4}, \quad z < p_1 \leq \sqrt{x} \quad \text{and} \quad p_1 \leq p_2 \leq x/(ap_1).$$

Also, we have $az^2 < ap_1^2 \leq ap_1 p_2 \leq x$; in particular,

$$a < x/z^2 \leq z < p_1, \quad \mathbf{1}_{\mathbb{B}}(a) = 1 \quad \text{and} \quad z < p_1 \leq \sqrt{x/a}.$$

Hence

$$T \leq \sum_{a \leq x/z^2} \mathbf{1}_{\mathbb{B}}(a) \sum_{\substack{z < p_1 \leq \sqrt{x/a} \\ p_1 \equiv 3 \pmod{4}}} \sum_{\substack{n_2 \leq x/(ap_1) \\ n_2 \equiv 3 \pmod{4}}} \mathbf{1}_{\mathbb{P}}(n_2) \cdot \mathbf{1}_{\mathbb{P}}(2kap_1 n_2 + 1).$$

We let (λ_{d_2}) and (λ_d) be any upper-bound sieves of level \hat{D} and “of beta type” (so that $\lambda_{d_2}, \lambda_d \in \{-1, 0, 1\}$). We note that as $\mathbf{1}_{\mathbb{P}}(n) \leq \sum_{\nu | n} \lambda_\nu$ ($\nu = d_2, d$) for every n , we have

$$\sum_{\substack{n_2 \leq x/(ap_1) \\ n_2 \equiv 3 \pmod{4}}} \mathbf{1}_{\mathbb{P}}(n_2) \cdot \mathbf{1}_{\mathbb{P}}(2ap_1 n_2 + 1) \leq \sum_{d_2, d} \lambda_{d_2} \lambda_d \sum_{\substack{n_2 \leq x/(ap_1) \\ n_2 \equiv 3 \pmod{4}, n_2 \equiv 0 \pmod{d} \\ 2ap_1 n_2 + 1 \equiv 0 \pmod{d_2}}} 1.$$

The three congruences in the last sum hold only if $(d_2, d) = (d_2, 2a) = (2, d) = 1$, so combining what we have so far, we obtain (for some residue $b \pmod{4d_2d}$),

$$\begin{aligned} T &\leq \sum_{a \leq x/z^2} \mathbf{1}_{\mathbb{B}}(a) \sum_{\substack{z < p_1 \leq \sqrt{x/a} \\ p_1 \equiv 3 \pmod{4}}} \sum_{\substack{(d_2, d)=1 \\ (d_2, 2a)=1 \\ (2, d)=1}} \lambda_{d_2} \lambda_d \sum_{\substack{n_2 \leq x/(ap_1) \\ n_2 \equiv b \pmod{4d_2d}}} 1 \\ &= \sum_{a \leq x/z^2} \mathbf{1}_{\mathbb{B}}(a) \sum_{\substack{z < p_1 \leq \sqrt{x/a} \\ p_1 \equiv 3 \pmod{4}}} \sum_{\substack{(d_2, d)=1 \\ (d_2, 2a)=1 \\ (2, d)=1}} \lambda_{d_2} \lambda_d \left\{ \frac{x}{4ap_1 d_2 d} + O(1) \right\}. \end{aligned}$$

The contribution of the O -term to the sum is $\ll \hat{D}^2 x/z \leq \hat{D}^2 z^2$. By a general result [13, Theorem 5.9] on the composition of linear sieves,

$$\sum_{\substack{(d_2, d)=1 \\ (d_2, 2a)=1 \\ (2, d)=1}} \frac{\lambda_{d_2} \lambda_d}{d_2 d} \leq \frac{4C + o(1)}{(\log \hat{D})^2} \frac{2a}{\phi(2a)} \left(\frac{2}{\phi(2)} \right) \leq \frac{16C + o(1)}{(\log \hat{D})^2} \frac{k}{\phi(k)} \frac{a}{\phi(a)},$$

where $o(1)$ denotes a quantity tending to zero as \hat{D} tends to infinity and⁴

$$C = \prod_{p \nmid 2a} (1 + (p-1)^{-2}) \leq \frac{1}{2} \prod_p (1 + (p-1)^{-2}) = 1.413 \dots$$

Thus, $16C + o(1) < 24$ if \hat{D} is sufficiently large, as we now assume. Gathering all of this, then using the fact that $\sum_{a \leq x/z^2} \mathbf{1}_{\mathbb{B}}(a)/\phi(a) \leq W(x/z^2)$ (cf. (3.7)) and the bound (3.10), we obtain that

$$\begin{aligned} T &\leq \frac{6x}{\phi(k)(\log \hat{D})^2} \sum_{a \leq x/z^2} \frac{\mathbf{1}_{\mathbb{B}}(a)}{\phi(a)} \sum_{\substack{z < p_1 \leq \sqrt{x} \\ p_1 \equiv 3 \pmod{4}}} \frac{1}{p_1} + O(\hat{D}^2 z^2) \\ &\leq \frac{3xW(x/z^2) \log(x/z^2)}{2\phi(k)(\log \hat{D})^2 \log z} \left(1 + O\left(\frac{1}{\log(x/z^2)} \right) \right) + O(\hat{D}^2 z^2). \end{aligned}$$

We want to exchange the factor $xW(x/z^2)/\phi(k)$ for $XV(z)$, where recall that $X := \pi(2kx+1)/\phi(8k)$. We have $x/(2\phi(k)) = X(\log x)(1 + O(1/\log x))$ by the prime number theorem. By (3.11) we have

$$W(x/z^2) = W(z) \left(\frac{\log(x/z^2)}{\log z} \right)^{1/2} \left(1 + O\left(\frac{1}{\log(x/z^2)} \right) \right),$$

and by (3.8) we have, with A_1 and A_3 being the constants defined in (3.9),

$$W(z) = \frac{1}{2} A_1 A_3 e^\gamma V(z) (\log z) \left(1 + O\left(\frac{1}{\log z} \right) \right).$$

Gathering once more we obtain

$$T \leq \frac{3}{2} A_1 A_3 e^\gamma XV(z) \frac{(\log x)(\log(x/z^2))^{3/2}}{(\log \hat{D})^2 (\log z)^{1/2}} \left(1 + O\left(\frac{1}{\log(x/z^2)} \right) \right) + O(\hat{D}^2 z^2).$$

We now set $\hat{D} := \sqrt{z}/x^\delta$. We have

$$x/z^2 < z, \quad \log z \asymp \log \hat{D} \asymp \log x, \quad \log(x/z^2) \asymp \delta \log x, \quad \hat{D}^2 z^2 = x^{1-2\delta}.$$

⁴The constant $\prod_p (1 + (p-1)^{-2}) = 2.826 \dots$ is known as Murata's constant.

It is therefore apparent that $T \ll \delta^{3/2} XV(z)$. To be more precise,

$$\frac{(\log x)(\log(x/z^2))^{3/2}}{(\log \hat{D})^2(\log z)^{1/2}} = (4\delta)^{3/2}(1/4 - 2\delta)^{-2}(1/2 - 2\delta)^{1/2} < 240\delta^{3/2}.$$

Finally, it is clear that $A_1 A_3 \leq \prod_p (1 + 1/(p(p-2)))$ (see the definition (3.9) of A_1 and A_3), and it is straightforward to verify that this product is less than $\prod_p (1 - p^{-2})^{-1} = \pi^2/6$. Hence

$$T \leq 60\pi^2 e^\gamma \delta^{3/2} XV(z) \{1 + O(1/(\delta \log x))\}.$$

A calculation shows that $60\pi^2 e^\gamma \delta^{3/2} = 0.0043 \dots$ (recall that $\delta = 1/3890$), and that by (3.13), $f(s) = 0.0341 \dots$. Hence (3.16). \square

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